JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **30**, No. 1, February 2017
http://dx.doi.org/10.14403/jcms.2017.30.1.15

MÖBIUS FUNCTIONS OF ORDER k IN FUNCTION FIELDS

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Abstract. We introduce the Möbius functions μ_k of order k and give the asymptotic formula for the summatory function associated to theses functions in function field case.

1. Introduction

In [1], Apostol introduced the following generalization of the Möbius function $\mu(n)$. Let k denote a fixed positive integer. The Möbius functions μ_k of order k is defined by $\mu_k(1) = 1$, $\mu_k(n) = 0$ if $p^{k+1}|n$ for some prime p, $\mu_k(n) = (-1)^r$ if $n = p_1^k p_2^k \cdots p_r^k \prod_{i>r} p_i^{a_i}$, $0 \le a_i < k$, $\mu_k(n) = 1$ otherwise. When $k = 1$, $\mu_k(n)$ is the usual Möbius function, $\mu_1(n) = \mu(n)$. Apostol established the following asymptotic formula ([1, Theorem 1]) for the summatory function $M_k(x) = \sum_{n\le x} \mu_k(n)$: for $k \ge 2$ and $x \ge 2$,

$$\sum_{n\le x} \mu_k(n) = A_k x + O\left(x^{1/k} \log x\right), \tag{1.1}$$

where A_k is the constant given by $A_k = \prod_p (1 - \frac{2}{p^k} + \frac{1}{p^{k+1}})$, the p runs over all primes. Suryanarayana [3] improved the O-estimate of the error term in (1.1) on the assumption of the Riemann hypothesis by proving the following: For $x \ge 3$,

$$\sum_{n\le x} \mu_k(n) = A_k x + O\left(x^{4k/(4k^2+1)} \exp\left(A \log x (\log\log x)^{-1}\right)\right),$$

Received August 10, 2016; Accepted December 16, 2016.
2010 Mathematics Subject Classification: Primary 11A25, 11N37, 11R58.
Key words and phrases: Möbius functions of order k, k-free polynomials, function field.
This work was supported by the research grant of Chungbuk National University in 2015.

where A being an absolute positive constant. In this paper we introduce the Möbius functions of order k and give the asymptotic formula for the summatory function associated to theses functions in function field case. Let $\mathbb{A} = \mathbb{F}_q[t]$ denote the polynomial ring over the finite field $\mathbb{F}_q$, where q is a power of an odd prime, and let $\mathbb{A}^+$ denote the set of monic polynomials in $\mathbb{A}$. For any integer $n \geq 0$, let $\mathbb{A}_n^+ = \{f \in \mathbb{A}^+ : \deg(f) = n\}$. In §2 we introduce the Möbius functions μ_k of order k in $\mathbb{A}^+$ and give the asymptotic formula for the summatory function $M_k(n) = \sum_{f\in\mathbb{A}_n^+} \mu_k(f)$ by using the analogue of Perron's formula in function fields. In §3 we discuss on the relation between the Möbius functions of order k and k-free polynomials.

We fix the following notations throughout the paper.

- $\mathcal{P}$:= the set of monic irreducible polynomials in $\mathbb{A}$.
- $\zeta_{\mathbb{A}}(s) = \frac{1}{1-q^{1-s}}$, the zeta function of $\mathbb{A}$.
- $\mathcal{Z}(u) = \frac{1}{1-qu}$, that is, $\mathcal{Z}(q^{-s}) = \zeta_{\mathbb{A}}(s)$.
- $|f| = q^{\deg(f)}$ for $0 \neq f \in \mathbb{A}$.

2. Möbius function of order k

We define an arithmetical function μ_k, the Möbius function of order k, as follows: For any $f \in \mathbb{A}^+$,

$$\mu_k(f) = \begin{cases} 1 & \text{if } f = 1, \\ 0 & \text{if } P^{k+1}|f \text{ for some } P \in \mathcal{P}, \\ (-1)^r & \text{if } f = P_1^k \cdots P_r^k \prod_{i>r} P_i^{a_i},\ 0 \leq a_i < k, \\ 1 & \text{otherwise.} \end{cases}$$

When $k = 1$, $\mu_k(f)$ is the usual Möbius function $\mu(f)$ on $\mathbb{A}^+$, i.e., $\mu_1(f) = \mu(f)$. It is easy to see that μ_k is a multiplicative function, that is, $\mu_k(fg) = \mu_k(f)\mu_k(g)$ whenever $(f, g) = 1$. Let $M_k(n) = \sum_{f\in\mathbb{A}_n^+} \mu_k(f)$ denote the summatory function associated to μ_k. When $k = 1$, the exact formula for $M_1(n)$ is given by (see [4, page 20])

$$M_1(n) = \begin{cases} 1 & \text{if } n = 0, \\ -q & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases} \tag{2.1}$$

For $k \geq 2$, we have the following asymptotic formula for the summatory function $M_k(n)$, which is a function field analogue of Apostol's Theorem (see [1, Theorem 1], [3, (3)]).

THEOREM 2.1. *Let $k \geq 2$ be an integer. For any $\epsilon > 0$, we have that as $n \to \infty$,*

$$M_k(n) = A_k q^n + O(q^{n\epsilon}),$$

where

$$A_k = \prod_{P \in \mathcal{P}} \left(1 - \frac{2}{|P|^k} + \frac{1}{|P|^{k+1}}\right).$$

Proof. Consider the generating function of $M_k(n)$:

$$\mathcal{M}_k(u) = \sum_{n=0}^{\infty} M_k(n) u^n = \sum_{f \in \mathbb{A}^+} \mu_k(f) u^{\deg(f)}.$$

By manipulating the Euler product, we have

$$\mathcal{M}_k(u) = \mathcal{Z}(u) G_k(u) = \frac{G_k(u)}{1 - qu},$$

where

$$G_k(u) = \prod_{P \in \mathcal{P}} \left(1 - 2u^{k \deg(P)} + u^{(k+1)\deg(P)}\right).$$

Note that $G_k(u)$ converges absolutely in the region $|u| < 1$, so that $\mathcal{M}_k(u)$ converges absolutely in the region $|u| < q^{-1}$. Using the Perron's formula, we have

$$\mathcal{M}_k(u) = \frac{1}{2\pi i} \oint_{|u| = q^{-1-\epsilon}} \frac{G_k(u)}{(1 - qu) u^{n+1}} du.$$

We enlarge the contour $|u| = q^{-1-\epsilon}$ to $|u| = q^{-\epsilon}$, and we encounter only one simple pole at $u = q^{-1}$. Hence, we have

(2.2)

$$\mathcal{M}_k(u) = \frac{1}{2\pi i} \oint_{|u| = q^{-\epsilon}} \frac{G_k(u)}{(1 - qu) u^{n+1}} du - \operatorname{Res}\left(\frac{G_k(u)}{(1 - qu) u^{n+1}}; u = q^{-1}\right).$$

Since $G_k(u)$ converges absolutely in the region $|u| < 1$, we have

$$\frac{1}{2\pi i} \oint_{|u| = q^{-\epsilon}} \frac{G_k(u)}{(1 - qu) u^{n+1}} du \ll q^{n\epsilon}. \tag{2.3}$$

The residue of $\frac{G_k(u)}{(1-qu)u^{n+1}}$ at $u = q^{-1}$ is given by

$$\operatorname{Res}\left(\frac{G_k(u)}{(1 - qu) u^{n+1}}; u = q^{-1}\right) = -A_k q^n. \tag{2.4}$$

By inserting (2.3) and (2.4) into (2.2), we get the result. $\square$

3. k-free polynomials

Let $k \geq 2$ be an integer. A polynomial $f \in \mathbb{A}^+$ is said to be *k-free* if $P^k \nmid f$ for any $P \in \mathcal{P}$. Let $\mathcal{Q}_k$ denote the set of k-free polynomials in $\mathbb{A}^+$, and let λ_k denote the characteristic function of $\mathcal{Q}_k$: for any $f \in \mathbb{A}^+$,

$$\lambda_k(f) = \begin{cases} 1 & \text{if } f \in \mathcal{Q}_k, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that λ_k is a multiplicative function, that is, $\lambda_k(fg) = \lambda_k(f)\lambda_k(g)$ whenever $(f,g) = 1$. Let $N_k(n) = \sum_{f \in \mathbb{A}_n^+} \lambda_k(f)$ be the summatory function associated to λ_k. When $k = 2$, we have (see [4, Proposition 2.3])

$$N_2(n) = \begin{cases} q^n & \text{if } n = 0 \text{ or } 1, \\ \frac{q^n}{\zeta_{\mathbb{A}}(2)} & \text{if } n \geq 2. \end{cases} \tag{3.1}$$

We have the following exact formula for the summatory function $N_k(n)$ for any $k \geq 2$, which is a generalization of (3.1) and a function field analogue of Gegenbauer's theorem (see [2, page 47]).

Theorem 3.1. *Let $k \geq 2$ be an integer. We have*

$$N_k(n) = \begin{cases} q^n & \textit{if } 0 \leq n \leq k-1, \\ \frac{q^n}{\zeta_{\mathbb{A}}(k)} & \textit{if } n \geq k. \end{cases}$$

Proof. Consider the generating function of $N_k(n)$:

$$\mathcal{N}_k(u) = \sum_{n=0}^{\infty} N_k(n)u^n = \sum_{f \in \mathbb{A}^+} \lambda_k(f)u^{\deg(f)}.$$

By manipulating the Euler product, we have

$$\mathcal{N}_k(u) = \prod_{P \in \mathcal{P}} \left(\frac{1 - u^{k\deg(P)}}{1 - u^{\deg(P)}} \right) = \frac{\mathcal{Z}(u)}{\mathcal{Z}(u^k)} = \frac{1 - qu^k}{1 - qu}.$$

Now by comparing the coefficients, we get the result. $\square$

From the definition of μ_k it follows that $\lambda_{k+1}(f) = |\mu_k(f)|$. By (3.1), we have that for $n \geq 2$,

$$\sum_{f \in \mathbb{A}_n^+} |\mu(f)| = \frac{q^n}{\zeta_{\mathbb{A}}(2)} \tag{3.2}$$

and, by Theorem 3.1, we have that for $k \geq 2$ and $n \geq k+1$,

$$\sum_{f \in \mathbb{A}_n^+} |\mu_k(f)| = \frac{q^n}{\zeta_{\mathbb{A}}(k+1)}. \tag{3.3}$$

Let $X_{k;n} = \{f \in \mathbb{A}_n^+ : \mu_k(f) = 1\}$ and $Y_{k;n} = \{f \in \mathbb{A}_n^+ : \mu_k(f) = -1\}$. By (2.1) and (3.2), we have that for $n \geq 2$,

$$\sharp X_{1;n} = \sharp Y_{1;n} = \frac{q^n}{2\zeta_{\mathbb{A}}(2)}.$$

Hence, the square-free polynomials with $\mu_f = 1$ occur with the same frequency as those with $\mu(f) = -1$. By Theorem 2.1 and (3.3), we have that for $k \geq 2$,

$$\sharp X_{k;n} = \frac{1}{2}\left(\frac{1}{\zeta_{\mathbb{A}}(k+1)} + A_k\right) q^n + O(q^{n\epsilon})$$

and

$$\sharp Y_{k;n} = \frac{1}{2}\left(\frac{1}{\zeta_{\mathbb{A}}(k+1)} - A_k\right) q^n + O(q^{n\epsilon}).$$

Hence, we see that among the $(k+1)$-free polynomials, $k > 1$, those for which $\mu_k(f) = 1$ occur asymptotically more frequently than those for which $\mu_k(f) = -1$; in particular, these sets of polynomials have, respectively, the densities

$$\frac{1}{2}\left(\frac{1}{\zeta_{\mathbb{A}}(k+1)} + A_k\right) \quad \text{and} \quad \frac{1}{2}\left(\frac{1}{\zeta_{\mathbb{A}}(k+1)} - A_k\right).$$

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